Relativistic Phase Invariance of Light

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1. The Problem

We will try to show that the difference in phase of a light beam between an emitter and comoving receiver is Lorentz invariant.

We will define the phase difference ϕ as the spatial path measured in units of wavelength, a real number.

$$\phi = \frac{\Delta L}{\lambda} \tag{1}$$

where ΔL is the spatial length of the path and λ is the wavelength of the light. All that is necessary is to show that the numerator and denominator in (1) transform in the same way. We are working in units where $c = 1$.

The distance between the receiver and emitter is that measured with a radar or laser ranging method. The wavelength of the signal whose phase we are calculating has been measured independently. The ratio ϕ is not directly observable and must be calculated by any observer.

2. Solution

Consider a light beam travelling from points P and Q with

$$P = \langle t_0,\ x_0,\ 0,\ 0\rangle^T, \quad Q = \langle t_0 + dl,\ x_0 + dl,\ 0,\ 0\rangle^T \tag{2}$$

dl is a positive constant. The distance between P and Q is

$$\Delta L = ((Q - P)[1]) = dl \tag{3}$$

We will now transform P, Q with a boost in the x-direction using the transformation

$$\Lambda = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

which gives

$$P^{'} = \Lambda P = \begin{bmatrix} x_0\,\beta\,\gamma + t_0\,\gamma & t_0\,\beta\,\gamma + x_0\,\gamma & 0 & 0 \end{bmatrix}^T \tag{5}$$

$$Q^{'} = \Lambda Q = \begin{bmatrix} (x_0 + dl)\ \beta\,\gamma + (t_0 + dl)\ \gamma & (t_0 + dl)\ \beta\,\gamma + (x_0 + dl)\ \gamma & 0 & 0 \end{bmatrix}^T \tag{6}$$

and the length

$$\Delta L' = ((Q' - P')[1]) = dl\ \gamma\,(1 + \beta) \tag{7}$$

The wavelength λ transforms with the Doppler shift formula

$$\lambda^{'} = \lambda\gamma\left(1+\beta\right) = \lambda\sqrt{\frac{1+\beta}{1-\beta}} \tag{8}$$

and so

$$\phi = \frac{\Delta L}{\lambda} = \frac{\Delta L^{'}}{\lambda^{'}} = \phi^{'} \tag{9}$$

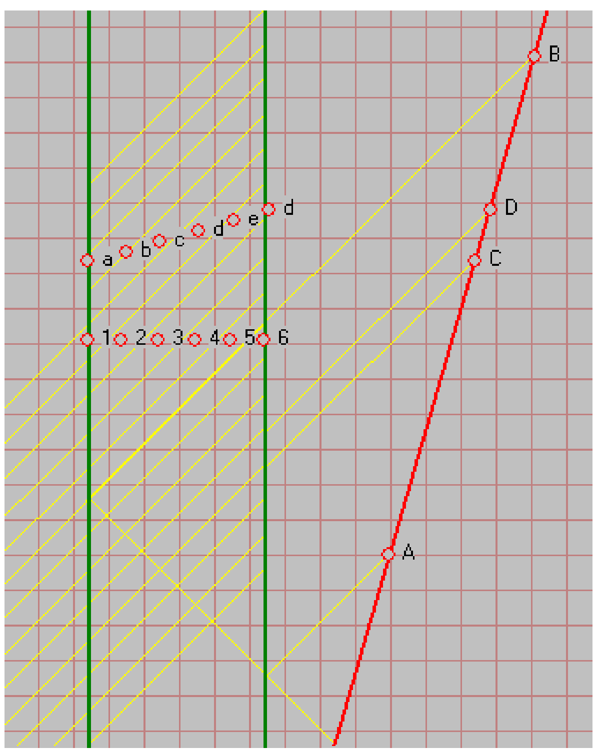
3. Conclusion

Clearly the phase difference, as defined in (1) is a Lorentz scalar. The results are invariant under spatial rotation and coordinate translation, so it is also a Poincare invariant.

4. A Cautionary Tale

I have elsewhere referred to the phase ratio ϕ as "the number of wavepeaks between the receiver and emitter" (N_p) and claimed it was an invariant. "DaleSpam", an SA in the Physics Forums, has pointed out that this definition is defective. It is certainly the case if we define N_p in a way that requires "a line of simultaneity". In this case N_p is frame-dependent and unobservable and probably meaningless. The situation is illustrated below.

The yellow worldlines are the wavepeaks of a continuous signal transmitted from the left-green worldline (WL) to the right-green WL. The red WL is an observer travelling away from the green observers.



The points labelled (1,2,3,4,5,6) are on a 'line of simultaneity' (LoS) of the green observers and those labelled (a,b,c,d,e,d) are on a LoS for the red worldline. Clearly the number of intersections with the worldlines of the wavepeaks and the two LoS are different and will depend on the relative velocity.

This does not mean there is a paradox. The events along the LoS are not directly observable. Any method of measuring the ratio ΔL and λ will yield the same result for ϕ. This is illustrated in the diagram where the ranging signal from the red worldline and the reflections are shown. The red observer measures the distance between the green observers as the half the proper time between events A and B. The time between wavepeaks is detected directly and is the proper interval CD. Clearly both of these measures has undergone the same Doppler shift, and so their ratio will yield the same result as the green frame calculation. This is true of any system of measurement that does not flout causality.

I'm grateful to "DaleSpam" for pointing out the problem. It emphasises the need to give clear, if possible operational definitions of the observables which I did not do with the "number of wavepeaks ..." idea.